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擬似双直交性理論とその応用

THEORY OF PSEUDO BIORTHOGONAL BASES AND ITS APPLICATION

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Abstract

This paper introduces the theory of pseudo biorthogonal bases (PBOB) which is the extension of the biorthonormal bases to linearly dependent over-complete systems. As an application of the PBOB, a generalized sampling theorem is derived.

1 Introduction

In order to provide redundant expansions of signals, the concept of a pseudo orthogonal basis (POB) was introduced in [2,3]. Although the POB uses larger number of elements than the dimension of the signal space for expansions, it preserves the same form as orthonormal expansion such as the Parseval's equality.

As it is possible to extend the concept of the orthonormal basis (ONB) to the concept of the biorthonormal basis (BONB), we extended the concept of POB to the concept of pseudo biorthogonal basis (PBOB) [4, 5, 6]. PBOB and POB have already been used to various applications such as signal restoration and the computerized tomography [7,8].

This paper reorganizes the theory of PBOB. The relationship between PBOB and the frame theory is discussed. Properties of the PBOB are analyzed in detail.

As an application of PBOB, a generalized sampling theorem is derived. It uses only finite number of sample points and provides the best approximation to the original function.

2 Definition of Pseudo Biorthogonal Basis

Let $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ be a set of $2M (M \geq N)$ elements in an N-dimensional Hilbert space H_N. If any element f in H_N can be expressed as

$$f = \sum_{m=1}^{M} \langle f, \phi_m^* \rangle \phi_m, \tag{1}$$

then $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ is said to be *a pseudo biorthogonal basis* in H_N or a PBOB for short. The set $\{\phi_m^* : 1 \leq m \leq M\}$ is called *a dual sequence* to $\{\phi_m : 1 \leq m \leq M\}$ and $\{\phi_m : 1 \leq m \leq M\}$ is called *a counter-dual sequence* to $\{\phi_m^* : 1 \leq m \leq M\}$.

Eq.(1) is the same expression of f in the form of a biorthonormal expansion. However, eq.(1) uses two sets of M elements $\{\phi_m : 1 \leq m \leq M\}$ and $\{\phi_m^* : 1 \leq m \leq M\}$ in an N-dimensional space H_N. Each of $\{\phi_m : 1 \leq m \leq M\}$ and $\{\phi_m^* : 1 \leq m \leq M\}$ is a linearly dependent set if $M > N$.

If $\phi_m^* = \phi_m$ for all m, the PBOB reduces to a pseudo orthogonal basis (POB). If $M = N$, it will be either an orthonormal basis (ONB) if $\phi_m^* = \phi_m$ for all m or a biorthonormal basis (BONB) if $\phi_m^* \neq \phi_m$. These relationships are summarized in Table 1.

Example 1 *Let α and β be arbitrary complex numbers. If we put*

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \phi_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2}$$

Table 1: Relationships among the different bases

	$\phi_m^* = \phi_m$	$\phi_m^* \neq \phi_m$
$M = N$	ONB: orthonormal basis	BONB: biorthonormal basis
$M \neq N$	POB: pseudo orthogonal basis	PBOB: pseudo biorthogonal basis

$$\phi_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \phi_2 = \begin{pmatrix} \alpha - 1 \\ \beta + 1 \end{pmatrix}, \phi_3 = \begin{pmatrix} -\alpha + 1 \\ -\beta \end{pmatrix}, \tag{3}$$

then, $\{\phi_m, \phi_m^* : 1 \leq m \leq 3\}$ *is a PBOB in* C^3.

Example 2 *Let* $\mathrm{H_M}$ *be an* M*-dimensional Hilbert space which includes* $\mathrm{H_N}$, *and* P_N *be an orthogonal projection operator from* $\mathrm{H_M}$ *onto* $\mathrm{H_N}$. *Let* $\{u_m, u_m^* : 1 \leq m \leq M\}$ *be a BONB in* $\mathrm{H_M}$. *If we put*

$$\phi_m = P_N u_m, \quad \phi_m^* = P_N u_m^* \qquad : 1 \leq m \leq M, \tag{4}$$

then $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ *is a PBOB in* $\mathrm{H_N}$.

Eq.(1) is equivalent to the following operator equation:

$$\sum_{m=1}^{M} (\phi_m \otimes \overline{\phi_m^*}) = I_N \tag{5}$$

where I_N is the identity operator on $\mathrm{H_N}$ and $(\cdot \otimes \bar{\cdot})$ is the Neumann-Schatten product defined by

$$(u \otimes \overline{v})h = \langle h, v \rangle u \tag{6}$$

for fixed u and v in $\mathrm{H_N}$ and for any h in $\mathrm{H_N}$.

Since $(u \otimes \overline{v})^* = (v \otimes \overline{u})$, eq.(5) implies that we can exchange $\{\phi_m : 1 \leq m \leq M\}$ with $\{\phi_m^* : 1 \leq m \leq M\}$ in eq.(1), i.e., it is true that for a PBOB

$$f = \sum_{m=1}^{M} \langle f, \phi_m \rangle \phi_m^*. \tag{7}$$

3 PBOB and Frame

Before going on with detailed discussions on PBOB, we shall show the relation between the concepts of *PBOB* and *frames.*

A set of elements $\{\phi_m : 1 \leq m \leq M\}$ in $\mathrm{H_N}$ is said to be *a frame* of $\mathrm{H_N}$ if there exist positive and finite constants A and B such that

$$A\|f\|^2 \leq \sum_{m=1}^{M} |\langle f, \phi_m \rangle|^2 \leq B\|f\|^2 \tag{8}$$

for every $f \in \mathrm{H_N}$. The numbers A and B are called *the frame bounds*[1,11].

For a frame $\{\phi_n : 1 \leq m \leq M\}$, if we define

$$S = \sum_{m=1}^{M} (\phi_m \otimes \overline{\phi_m}), \tag{9}$$

$$\phi_m^* = S^{-1}\phi_m, \tag{10}$$

then eqs.(7) and (1) hold. Hence, a frame together with $\{\phi_m^* : 1 \leq m \leq M\}$ in eq.(10) is a PBOB. The operator S is called the frame operator, which is nonsingular. The set $\{\phi_m^* : 1 \leq m \leq M\}$ is called the standard dual frame.

The converse also holds as follows. For a set $\{\phi_n : 1 \leq m \leq M\}$ which spans $\mathrm{H_N}$, there always exists a dual sequence $\{\phi_n^* : 1 \leq m \leq M\}$ which will be shown in Section 6. Hence, we have

Theorem 1 *If a set $\{\phi_n : 1 \leq m \leq M\}$ spans $\mathrm{H_N}$, then it forms a frame.*

Proof. Let $\{\phi_n^* : 1 \leq m \leq M\}$ be a dual sequence of $\{\phi_n : 1 \leq m \leq M\}$. If we put

$$A = (\sum_{m=1}^{M} \|\phi_m^*\|^2)^{-1}, \tag{11}$$

$$B = \sum_{m=1}^{M} \|\phi_m\|^2, \tag{12}$$

then it follows from eq.(7) and the Schwarz inequality that

$$\begin{aligned}
\|f\|^2 &= \|\sum_{m=1}^{M} \langle f, \phi_m\rangle \phi_m^*\|^2 \\
&\leq (\sum_{m=1}^{M} |\langle f, \phi_m\rangle| \; \|\phi_m^*\|)^2 \\
&\leq (\sum_{m=1} |\langle f, \phi_m\rangle|^2) \sum_{m=1}^{M} \|\phi_m^*\|^2.
\end{aligned}$$

Hence, applying the Schwarz inequality again, we have

$$A\|f\|^2 \leq \sum_{m=1}^{M} |\langle f, \phi_m\rangle|^2 \leq \sum_{m=1}^{M} \|f\|^2\|\phi_m\|^2 = B\|f\|^2.$$

This establishes the theorem. ■

PBOBs and frames are essentially the same for finite dimensional spaces as shown above. However, if the name 'PBOB' is given to the set $\{\phi_m, \phi_m^*\}$ in eq.(1), the name 'frame' is given only to the set $\{\phi_m\}$ in eq.(7). This means that PBOB emphasizes the fact that each PBOB sequence $\{\phi_m\}$ has a dual sequence $\{\phi_m^*\}$.

The concept of frame was proposed by Duffin and Schaeffer in 1952 in terms of nonharmonic Fourier series [1]. It is only after a long period of silence that Young's book in 1980 illuminated again the concept of frame. And this was done also in terms of nonharmonic Fourier series [11]. However, in 1973, during this period of silence, Iijima and Ogawa proposed a concept of POB for redundant expansions of signals independently of Duffin and Schaeffer's work [2,4]. Ogawa also extended the concept of POB to the concept of PBOB in 1978 [4,5,6]. He established in detail general properties of frames in terms of PBOB. PBOB was applied to various problems such as signal restoration, computerized tomography, and neural network learning problems [7,8]

4 Characterization of PBOB

In this section we provide a characterization of PBOBs. The following notations are fundamental in our theory of PBOB. For a given set of $2M(\geq N)$ elements $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ in $\mathrm{H_N}$, let $\mathrm{H_M}$ be any fixed M-dimensional Hilbert space. Let $\{\varphi_n : 1 \leq n \leq N\}$ and $\{\varphi'_m : 1 \leq m \leq M\}$ be any fixed orthonormal bases in $\mathrm{H_N}$ and $\mathrm{H_M}$, respectively. Let us define

$$
\begin{aligned}
U &= \sum_{m=1}^{M} (\varphi'_m \otimes \overline{\phi_m}), && (13)\\
V &= \sum_{m=1}^{M} (\varphi'_m \otimes \overline{\phi_m^*}), && (14)\\
W &= UV^*, && (15)\\
u_{m,n} &= \langle \varphi_n, \phi_m \rangle, && (16)\\
v_{m,n} &= \langle \varphi_n, \phi_m^* \rangle, && (17)\\
w_{m,n} &= \langle \phi_n^*, \phi_m \rangle, && (18)
\end{aligned}
$$

The operators U and V play a central role in the theory of PBOB. It follows from eqs.(13) and (14) that

$$
U^*V = \sum_{m=1}^{M} (\phi_m \otimes \overline{\phi_m^*}). \qquad (19)
$$

Though the operators U and V are defined by using $\{\varphi'_m : 1 \leq m \leq M\}$ in $\mathrm{H_M}$, eq.(19) means that U^*V is independent of both the choice of space $\mathrm{H_M}$ and $\{\varphi'_m : 1 \leq m \leq M\}$ in $\mathrm{H_M}$. Since $\{\varphi'_m : 1 \leq m \leq M\}$ is an orthonormal basis in $\mathrm{H_M}$, it is true that

$$
\phi_m = U^*\varphi'_m, \quad \phi_m^* = V^*\varphi'_m. \qquad (20)
$$

Lemma 1 *The following hold.*

$$
\begin{aligned}
U &= \sum_{m=1}^{M} \sum_{n=1}^{N} u_{m,n} (\varphi'_m \otimes \overline{\varphi_n}), && (21)\\
V &= \sum_{m=1}^{M} \sum_{n=1}^{N} v_{m,n} (\varphi'_m \otimes \overline{\varphi_n}), && (22)\\
W &= \sum_{m=1}^{M} \sum_{n=1}^{N} w_{m,n} (\varphi'_m \otimes \overline{\varphi_n}), && (23)
\end{aligned}
$$

Proof. It follows from eqs.(16) and (20) that

$$
u_{m,n} = \langle \varphi_n, \phi_m \rangle = \langle \varphi_n, U^*\varphi'_m \rangle = \langle U\varphi_n, \varphi'_m \rangle.
$$

Then, eq.(21) holds. The remaining parts can be derived similarly. ■

Eqs.(21) and (22) state that $(u_{m,n})$ and $(v_{m,n})$ are the matrix representations of the operators U and V, respectively, with respect to the ONBs $\{\varphi_n : 1 \leq n \leq N\}$ in $\mathrm{H_N}$ and $\{\varphi'_m : 1 \leq m \leq M\}$ in $\mathrm{H_M}$. Similarly, eq.(23) means that $(w_{m,n})$ is the matrix representation of the operator W with respect to the ONB $\{\varphi'_m : 1 \leq m \leq M\}$ in $\mathrm{H_M}$.

Now we can characterize a PBOB as follows.

Theorem 2 *The following statements are mutually equivalent.*

(i) $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ *is a PBOB, i.e., eq.(1) holds.*

(ii) $U^*V = I_N.$ (24)

(iii) $W^2 = W,$ (25)

$N(U) = N(V) = \{0\}.$ (26)

(iv) $\langle Uf, Vg\rangle = \langle f, g\rangle.$ (27)

(v) $\langle f, g\rangle = \sum_{m=1}^{M} \langle f, \phi_m\rangle \overline{\langle g, \phi_m^*\rangle}.$ (28)

(vi) $\sum_{p=1}^{M} \overline{u_{p,m}} v_{p,n} = \delta_{m,n} \quad : 1 \leq m, n \leq N.$ (29)

Proof. (i)$\leftrightarrow$(ii): It is clear from eqs.(19) and (5).

(ii)$\rightarrow$(iii): It follows from (ii) that

$$V^*U = I_N. \tag{30}$$

Then, eq.(15) yields

$$W^2 = UV^*UV^* = UI_NV^* = W,$$

which implies eq.(25). Since eq.(30) yields

$$\{0\} \subset N(U) \subset N(V^*U) = N(I_N) = \{0\},$$

it holds that $N(U) = \{0\}$. In the similar way, eq.(24) yields $N(V) = \{0\}$.

(iii)$\rightarrow$(ii): It follows from eqs.(25) and (15) that

$$U(V^*U - I_N)V^* = 0. \tag{31}$$

Since eq.(26) means $R(V*) = \mathrm{H_N}$, eq.(31) yields

$$U(V^*U - I_N) = 0. \tag{32}$$

Since $N(U) = \{0\}$, eq.(32) yields eq.(30), which implies (ii).

(ii)$\rightarrow$(iv): It follows from eq.(24) that $\langle Uf, Vg\rangle = \langle f, U^*Vg\rangle = \langle f, g\rangle$.

(iv)$\rightarrow$(ii): Since eq.(27) yields $\langle f, (U^*V - I_N)g\rangle = 0$, (ii) holds.

(iv)$\leftrightarrow$(v): Since eqs.(13) and (14) yield

$$Uf = \sum_{m=1}^{M} \langle f, \phi_m\rangle \varphi'_m, \tag{33}$$

$$Vg = \sum_{m=1}^{M} \langle g, \phi_m^*\rangle \varphi'_m, \tag{34}$$

we have

$$\langle Uf, Vg\rangle = \sum_{m=1}^{M} \langle f, \phi_m\rangle \overline{\langle g, \phi_m^*\rangle}, \tag{35}$$

since $\{\varphi'_m : 1 \leq m \leq M\}$ is an orthonormal basis in $\mathrm{H_M}$. This establishes (iv)$\leftrightarrow$(v).

(ii)$\leftrightarrow$(vi): It is clear from Lemma 1. ■

Eq.(24) is the most important property of the PBOB. That is, U^* is a left inverse of V. Eq.(25) means that W is an oblique projection operator in $\mathrm{H_M}$. More precisely, we have

Corollary 1 *If a set $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}f$ is a PBOB, then W is an oblique projection operator onto $R(U)$ along the direction of $R(V)^{\perp}$.*

Proof. Since $N(V) = \{0\}$, it holds that $R(V^*) = \mathrm{H_N}$. Then, eq.(15) yields

$$R(W) = R(UV^*) = UR(V^*) = U\mathrm{H_N} = R(U),$$

and hence $R(W) = R(U)$. Since $N(U) = \{0\}$, we have $U^{\dagger}U = I_N$. Then, eq.(15) yields

$$V^* = I_N V^* = U^{\dagger}UV^* = U^{\dagger}W.$$

Hence, we have

$$V^* = U^{\dagger}W. \tag{36}$$

It follows from eqs.(15) and (36) that

$$N(W) = N(UV^*) \supset N(V^*) = N(U^{\dagger}W) \supset N(W).$$

Hence, $N(W) = N(V^*) = R(V)^{\perp}$. ■

From Corollary 1 and Lemma 1, we have

Corollary 2 *If a set $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ is a PBOB, then $(w_{m,n})$ is an oblique projection matrix onto the range of $(u_{m,n})$ along the direction of the orthogonal complement of the range of $(v_{m,n})$.*

Eq.(28) is the extension of the Parseval's equality for a BONB. We can see from eq.(35) that the Parseval's equality (28) is only another expression of eq.(27). Putting g=f in Theorem 1 yields

Corollary 3 *The following statements are mutually equivalent.*

(i) $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ *is a PBOB.*

(ii) $$\langle Uf, Vf\rangle = \|f\|^2. \tag{37}$$

(iii) $$\|f\|2 = \sum_{m=1}^{M} \langle f, \phi_m\rangle \overline{\langle f, \phi_m^*\rangle}. \tag{38}$$

When $\mathrm{H_N}$ is a functional space, the following theorem is useful.

Theorem 3 *Let $\mathrm{H_N}$ be an N-dimensional reproducing kernel Hilbert space. Let $K(x, x')$ be a reproducing kernel of $\mathrm{H_N}$. A set $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ in $\mathrm{H_N}$ is a PBOB if and only if*

$$\sum_{m=1}^{M} \phi_m^*(x)\overline{\phi_m(x')} = K(x, x'). \tag{39}$$

Proof. Let us denote the left-hand side of eq.(39) by $H(x, x')$ temporarily. For any fixed x', $H(x, x')$ belongs to $\mathrm{H_N}$. Furthermore, we have for any fixed x' and any f in $\mathrm{H_N}$

$$\begin{aligned}\langle f(\cdot), H(\cdot, x')\rangle &= \langle f(\cdot), \sum_{m=1}^{M} \phi_m^*(\cdot)\overline{\phi_m(x')}\rangle \\ &= \sum_{m=1}^{M} \langle f, \phi_m^*\rangle \phi_m(x'),\end{aligned}$$

and hence

$$\langle f(\cdot), H(\cdot, x')\rangle = \sum_{m=1}^{M} \langle f, \phi_m^* \rangle \phi_m(x')$$

This implies the theorem. ■

The right-hand side of eq.(39) is independent of both the number of elements of the PBOB and the choise of PBOB itself. That is, the left-hand side of the equation is a kind of invariant of PBOB.

5 Properties of PBOB

As we can see from eqs.(28), (38), and (39), the Parseval's equality as well as the expression of the reproducing kernel by using the PBOB has the same form with respect to the BONB. This section focus on such form preservation properties of PBOB. Let $B(\mathrm{H_N})$ be a set of all bounded linear operators on $\mathrm{H_N}$.

Theorem 4 *Let* $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ *be a PBOB in* $\mathrm{H_N}$. *For any* $A \in B(\mathrm{H_N})$

$$A = \sum_{m=1}^{M} \sum_{n=1}^{M} \langle A\phi_n, \phi_m^* \rangle (\phi_m \otimes \overline{\phi_n^*}) \tag{40}$$

Proof. It follows from eqs.(5) and (1) that

$$\begin{aligned} A &= AI_N \\ &= A\sum_{n=1}^{M} (\phi_n \otimes \overline{\phi_n^*}) \\ &= \sum_{n=1}^{M} ([\sum_{m=1}^{M} \langle A\phi_n, \phi_m^* \rangle \phi_m] \otimes \overline{\phi_n^*}), \end{aligned}$$

which implies eq.(40). ■

Eq.(40) says that the Neumann-Schatten product expression of A in $B(\mathrm{H_N})$ by using the PBOB has the same form with respect to the BONB.

The expressions of the Schmidt norm, the Schmidt inner product, and the trace of an operator by using the PBOB have also the same form with respect to the BONB as follows:

Theorem 5 *Let* $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ *be a PBOB in* $\mathrm{H_N}$. *For any* $A, B \in B(\mathrm{H_N})$, *we have*

$$\langle A, B\rangle = \sum_{m=1}^{M} \langle A\phi_m^*, B\phi_m \rangle, \tag{41}$$

$$\|A\|_2^2 = \sum_{m=1}^{M} \langle A\phi_m^*, A\phi_m \rangle, \tag{42}$$

$$\mathrm{tr}(A) = \sum_{m=1}^{M} \langle A\phi_m^*, \phi_m \rangle. \tag{43}$$

Proof. It follows from eq.(5) that

$$\begin{aligned} \langle A, B\rangle &= \langle A, BI_N\rangle \\ &= \langle A, B\sum_{m=1}^{M} (\phi_m \otimes \overline{\phi_m^*})\rangle \end{aligned}$$

$$= \sum_{m=1}^{M} \langle A, (B\phi_m) \otimes \overline{\phi_m^*} \rangle$$
$$= \sum_{m=1}^{M} \langle A\phi_m^*, B\phi_m \rangle,$$

which implies eq.(41). The remaining is clear from eq.(41). ■

Theorem 6 *Let $\{\phi_m, \phi_m^* : 1 \le m \le M\}$ be a PBOB in $\mathrm{H_N}$. Then we have*

$$\mathrm{tr}(W) = \sum_{m=1}^{M} \langle \phi_m^*, \phi_m \rangle = N. \tag{44}$$

Proof. It follows from eqs.(15) and (20) that

$$\begin{aligned} \mathrm{tr}(W) &= \sum_{m=1}^{M} \langle W\varphi_m', \varphi_m' \rangle = \sum_{m=1}^{M} \langle UV^*\varphi_m', \varphi_m' \rangle \\ &= \sum_{m=1}^{M} \langle V^*\varphi_m', U^*\varphi_m' \rangle = \sum_{m=1}^{M} \langle \phi_m^*, \phi_m \rangle, \end{aligned}$$

and hence

$$\mathrm{tr}(W) = \sum_{m=1}^{M} \langle \phi_m^*, \phi_m \rangle. \tag{45}$$

When $\{\phi_m, \phi_m^* : 1 \le m \le M\}$ is a PBOB in $\mathrm{H_N}$, putting $A = I_N$ in eq.(43) yields

$$\sum_{m=1}^{M} \langle \phi_m^*, \phi_m \rangle = N.$$

This implies eq.(44) because of eq.(45). ■

The right-hand side of eq.(44) is independent of not only the number of elements of the PBOB but also the chosen PBOB itself. That is, the left-hand sides of the equation is invariant of the choice of PBOB.

Let $\{\phi_m, \phi_m^* : 1 \le m \le M\}$ be a PBOB in $\mathrm{H_N}$. If the inner products $\langle \phi_m^*, \phi_m \rangle = \langle \phi_n^*, \phi_n \rangle$ for all m and n, then it is called *a pseudo biorthonormal basis* or a PBONB for short. It follows from eq.(44) that

Corollary 4 *Let $\{\phi_m, \phi_m^* : 1 \le m \le M\}$ be a PBONB in $\mathrm{H_N}$. It holds that*

$$\langle \phi_m^*, \phi_m \rangle = \frac{N}{M} \qquad : 1 \le m \le M. \tag{46}$$

The right-hand side of eq.(46) is independent of the choice of PBOB. It depends only on the number of elements of the PBOB.

6 Construction of PBOB

Given a set $\{\phi_m : 1 \le m \le M\}$ which spans the whole space $\mathrm{H_N}$, we shall give general methods for the construction of a dual sequence $\{\phi_m^* : 1 \le m \le M\}$.

Theorem 7 *Let* $\mathrm{H_M}$ *be any fixed* M*-dimensional Hilbert space and* $\{\varphi'_m : 1 \leq m \leq M\}$ *be any fixed orthonormal basis in* $\mathrm{H_M}$*. Let* U *be an operator defined by eq.(13). Let* T *be any fixed left inverse of* U*. If we put*

$$\phi^*_m = T\varphi'_m, \tag{47}$$

then $\{\phi_m, \phi^*_m : 1 \leq m \leq M\}$ *is a PBOB in* $\mathrm{H_N}$*. All dual bases can be constructed by changing the left inverse* T*.*

Proof. Eq.(24) is equivalent to eq.(30). It follows from eqs.(14) and (47) that

$$V = \sum_{m=1}^{M} (\varphi'_m \otimes \overline{\phi^*_m}) = \sum_{m=1}^{M} (\varphi'_m \otimes \overline{\varphi'_m})T^* = T^*, \tag{48}$$

and hence $V^* = T$. Since T is a left inverse of U, we have $V^*U = TU = I_N$. Then $\{\phi_m, \phi^*_m : 1 \leq m \leq M\}$ is a PBOB in $\mathrm{H_N}$ because of eq.(30). The remaining of the theorem is clear from Theorem 2. ■

A general form of the left inverse of U is given as

$$T = U^\dagger + Y(I_M - UU^\dagger), \tag{49}$$

where Y is an arbitrary operator from $\mathrm{H_M}$ to $\mathrm{H_N}$. Another general form of the left inverse of U is given as

$$T = U^\dagger W, \tag{50}$$

where W is an arbitrary oblique projection operator onto $R(U)$. Note that eq.(50) is nothing but eq.(36) because $V^* = T$. By changing the operator Y or W, we can construct all dual sequences by using eq.(47).

The following method essentially needs only the left inverse of a matrix even when $\mathrm{H_N}$ is a space of functions.

Theorem 8 *Let* $\{\varphi_n : 1 \leq n \leq N\}$ *be any fixed orthonormal basis in* $\mathrm{H_N}$*. Let* $(u_{m,n})$ *be an* $M \times N$ *matrix defined by eq.(16). Let* $(t_{m,n})$ *be a left inverse of the matrix* $(u_{m,n})$*:*

$$\sum_{p=1}^{M} t_{n,p} u_{p,m} = \delta_{m,n} \qquad : 1 \leq m, n \leq N. \tag{51}$$

If we put

$$\phi^*_m = \sum_{n=1}^{N} t_{n,m} \varphi_n, \tag{52}$$

then $\{\phi_m, \phi^*_m : 1 \leq m \leq M\}$ *is a PBOB in* $\mathrm{H_N}$*. We can use this method to construct any PBOB.*

Proof. It follows from eqs.(17) and (52) that

$$v_{m,n} = \langle \varphi_n, \phi^*_m \rangle = \overline{t_{m,n}}, \tag{53}$$

Eqs.(53) and (51) yield eq.(29). This implies that $\{\phi_m, \phi^*_m : 1 \leq m \leq M\}$ is a PBOB in $\mathrm{H_N}$. The remaining of the theorem is clear from Theorem 2. ■

7 Sufficiency of PBOB

Let $\{\phi_m, \phi_m^* : 1 \le m \le M\}$ be a PBOB in H_N. If $M > N$, the set $\{\phi_m : 1 \le m \le M\}$ is linearly dependent. Hence, f in H_N can be expressed in infinitely many ways as

$$f = \sum_{m=1}^{M} a_m \phi_m. \tag{54}$$

That is, we have infinitely many sets of expansion coefficients $\{am : 1 \le m \le M\}$. A basic question on PBOB is that can all expansion coefficients be expressed as inner products of f and elements of a dual sequence as

$$a_m = \langle f, \phi_m^* \rangle \qquad : 1 \le m \le M. \tag{55}$$

We say that the PBOB is *sufficient* if eq.(54) and (54) holds.

If $f = 0$, then $\langle f, \phi_m^* \rangle = 0$ for all m. However, if $M > N$, there exist a set $\{a_m : 1 \le m \le M\}$ which includes non-zero elements even if $f = 0$. Therefore, the concept of "sufficiency of PBOB" makes sense when eq.55 holds for any $f \neq 0$.

Theorem 9 *Let $\{\phi_m : 1 \le m \le M\}$ be a set which spans the whole space H_N. Let f be a non-zero element of H_N. Let $\{a_m : 1 \le m \le M\}$ be any fixed expansion coefficients of f such that eq.(54) holds. There exists a dual sequence $\{\phi_m^* : 1 \le m \le M\}$ such that eq.(55) holds.*

Proof. Let $\{e_m : 1 \le m \le M\}$ be the standard basis in C^M and U be an operator defined by

$$U = \sum_{m=1}^{M} (e_m \otimes \overline{\phi_m}). \tag{56}$$

Let

$$f_1 = \frac{f}{||f||}, \qquad f_2 = \frac{f}{||f||^2}. \tag{57}$$

and P_1 be the orthogonal projection operator onto the orthogonal complement of the one-dimensional subspace spanned by f. Finally, let

$$\phi_m^* = \overline{a_m} f_2 + P_1 U^\dagger e_m. \tag{58}$$

We shall show the set $\{\phi_m^* : 1 \le m \le M\}$ meets the requirment of Theorem 9. The set $\{U^\dagger e_m : 1 \le m \le M\}$ is a dual sequence to $\{\phi_m : 1 \le m \le M\}$ because of eq.(49) and Theorem 7. Eqs.(58), (54), and (57) yield

$$\begin{aligned}
\sum_{m=1}^{M} \phi_m \otimes \overline{\phi_m^*} &= \sum_{m=1}^{M} \phi_m \otimes \overline{(\overline{a_m} f_2 + P_1 U^\dagger e_m)} \\
&= [(\sum_{m=1}^{M} a_m \phi_m) \otimes \overline{f_2}] + (\sum_{m=1}^{M} \phi_m \otimes \overline{U^\dagger e_m}) P_1 \\
&= (f \otimes \overline{f_2}) + P_1 \\
&= (f_1 \otimes \overline{f_1}) + [I_N - (f_1 \otimes \overline{f_1})] \\
&= I_N,
\end{aligned}$$

which implies that $\{\phi_m^* : 1 \le m \le M\}$ is a dual sequence to $\{\phi_m : 1 \le m \le M\}$. Eqs.(58) and (57) yield

$$\langle f, \phi_m^* \rangle = \langle f, \overline{a_m} f_2 + P_1 U^\dagger e_m \rangle = a_m \langle f, f_2 \rangle = a_m,$$

which implies eq.(55). ■

8 Pseudo Biorthogonal Bases of Type O and Type L

Let $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ be a PBOB in $\mathrm{H_N}$. In the previous sections, we discussed general properties of the PBOB which are independent of the choice of dual sequence. Among all dual bases, there exist very interesting classes. These will be discussed in this section.

Let us consider the operator W defined by eq.(15). For a PBOB, it is in general an oblique projection operator as shown in Theorem 2. If W is an orthogonal projection operator, then the PBOB is said to be *a pseudo biorthogonal basis of type O* or O-PBOB for short. The set $\{\phi_m^* : 1 \leq m \leq M\}$ is called *a dual sequence of type O* or an O-dual sequence to $\{\phi_m : 1 \leq m \leq M\}$. The appellation "O" refers to "orthogonal".

Let us consider the matrix $(w_{m,n})$ defined by eq.(18). Eq.(23) shows that the matrix $(w_{m,n})$ is a representation of the operator W, and $(w_{m,n})$ is an oblique projection matrix for a general PBOB. An oblique projection matrix becomes an orthogonal projection matrix if and only if it is Hermitian. Then, we have

Theorem 10 *A PBOB $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ is type O if and only if the matrix $(w_{m,n})$ is Hermitian.*

It turns out from Theorem 10 that the ONB, the BONB, and the POB are all type O. Furthermore, the PBOB given in Example 1 is an O-PBOB if and only if $\alpha = 2/3$ and $\beta = -1/3$.

For a PBOB $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ in $\mathrm{H_N}$, if there exists a linear operator A such that

$$\phi_m^* = A\phi_m \qquad : 1 \leq m \leq M, \tag{59}$$

then $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ is said to be *a pseudo biorthogonal basis of type L* or L-PBOB for short. The set $\{\phi_m^* : 1 \leq m \leq M\}$ is called *a dual sequence to type L* or an L-dual sequence to $\{\phi_m : 1 \leq m \leq M\}$. The appellation "L" refers to "linear".

Since A is a linear operator on $\mathrm{H_N}$, it can be determined by using only N linearly independent systems. Eq.(59), however, gives $M(M \geq N)$ number of conditions. Hence, the operator A does not exist in general, and the only special PBOB for which it exists is called an L-PBOB. The ONB, the BONB, and the POB are all type L. Furthermore, the PBOB given in Example 1 is an L-PBOB if and only if $\alpha = 2/3$ and $\beta = -1/3$, which is the same condition for the O-PBOB.

Theorem 11 *Let $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ be a PBOB in* $\mathrm{H_N}$. *The following statements are mutually equivalent.*

(i) $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ *is an O-PBOB.*

(iii) $R(U) = R(V)$. (60)

(iv) $\{\phi_m, \phi_m^* : 1 \leq m \leq M\}$ *is an L-PBOB.*

(vi) $\phi_m^* = (U^* U)^{-1} \phi_m$. (61)

(vii) $\phi_m^* = U^\dagger \varphi_m'$. (62)

(viii) $V^* = U^\dagger$. (63)

Proof. (i) $\leftrightarrow$ (iii): It is clear from Corollary 1.

(i) $\leftrightarrow$ (viii): Since (i) yields that $W = UU^\dagger$, it follows from eq.(36) that

$$V^* = U^\dagger W = U^\dagger U U^\dagger = U^\dagger.$$

which implies (i) $\rightarrow$ (viii). The converse is clear from eq.(15).

(viii) $\leftrightarrow$ (vii): It is clear from eq.(20).

(vii) $\leftrightarrow$ (vi): It is clear from eq.(20) and $U^\dagger = (U^*U)^{-1}U^*$.

(vi) $\leftrightarrow$ (iv): Let A be a linear operator such that eq.(59) holds. It follows from eq.(13) that

$$U^*U = \sum_{m=1}^{M} \phi_m \otimes \overline{\phi_m}. \tag{64}$$

Eqs.(59), (64), and (5) yield

$$A(U^*U) = I_N.$$

Since $N(U^*U) = N(U) = \{0\}$, U^*U is nonsingular and we have

$$A = (U^*U)^{-1}, \tag{65}$$

which implies (iv) $\rightarrow$ (vi). The converse is clear. ■

Eq.(61) guarantees existence and uniqueness of the L-dual sequence. Hence, we have

Theorem 12 *For any set $\{\phi_m : 1 \le m \le M\}$ which spans the whole space $\mathrm{H_N}$, there always exist an O-dual sequence and an L-dual sequence . They are uniquely determined and they are the same.*

The O-PBOB gives the minimal norm expansion coefficients in eq.(54) as follows. Let a be an M-dimensional vector consisting of the expansion coefficients in eq.(54). Let $\hat{f}$ be an M-dimensional vector consisting of the expansion coefficients of f with respect to the O-dual sequence $\{\phi_m^* : 1 \le m \le M\}$:

$$\hat{f} = \sum_{m=1}^{M} \langle f, \phi_m^* \rangle e_m. \tag{66}$$

Then, we have

Theorem 13 *Let $\{\phi_m, \phi_m^* : 1 \le m \le M\}$ be an O-PBOB. For any vector a which satisfies eq.(54), we have*

$$||\hat{f}|| \le ||a||. \tag{67}$$

The equality holds if and only if $a = \hat{f}$.

Proof. If we choose C^M as $\mathrm{H_M}$ and e_m as φ'_m in eqs.(13) and (14), then it holds that

$$U = \sum_{m=1}^{M} (e_m \otimes \overline{\phi_m}), \tag{68}$$

$$V = \sum_{m=1}^{M} (e_m \otimes \overline{\phi_m^*}), \tag{69}$$

Eq.(54) can be expressed as

$$U^* a = f, \tag{70}$$

because of eq.(68). Eq.(70) has a unique minimal norm solution, which is given by $(U^*)^\dagger f$. Since it follows from eq.(63) that $(U^*)^\dagger = V$, eqs.(69) and (66) yield $(U^*)^\dagger f = Vf = \hat{f}$. This implies the theorem. ■

Theorem 11 provides methods of construction of O-dual sequence or L-dual basis $\{\phi_m^* : 1 \leq m \leq M\}$ for a given set $\{\phi_m : 1 \leq m \leq M\}$ which spans $\mathrm{H_N}$. For example, Theorem 11(i) provides the method which is given by Theorem 7 with the orthogonal projection operator W in eq.(50). Eq.(62) provides the method which is also given by Theorem 7 with $Y = 0$ in eq.(49). Since U^*U is the frame operator as indicated by eq.(64) and (9), eq.(61) provides the method given by eqs.(9) and (10). That means that the standard dual frame is nothing but the L-dual sequence.

9 General Sampling Theorem

In this section, we shall develop a general sampling theorem by using the theorey of PBOB. Since the sampling theorem was introduced to communications theory by Someya [10] and Shannon [9] in 1949, it has been extended in various directions. The extensions include, for example, sampling with nonuniformly spaced sample points, sampling for the band-pass functions instead of the usual low-pass functions, sampling for functions of more than one variable, and sampling for general integral transforms.

Two view points are considered in sampling theories. The first is the point of view of the general Fourier expansion. Almost all results presented so far have been done from this point of view.

In practical applications, we can use only finite number of sample values. That means that a direct application of the conventional sampling theorem causes the so-called truncation error. It leads us to the point of view of the function approximation. That is the second viewpoint.

The conventional sampling theorem needs exact sample values of a function, which are impossible to obtain in practical applications. Only blurred values which come out from some measuring equipment are available. The sampling theorem for such blurred samples is called a sampling with real pulse. On the other hand, the traditional sampling theorem is called a sampling with ideal pulse.

In this section we propose a generalized sampling theorem which uses a finite number of sample values of the original or blurred function. Two viewpoints mentioned above are unified.

Let H be an infinite or finite dimensional functional Hilbert space consisting of complex (or real) valued functions $f(x)$ defined on a one- or multi-dimensional domain D. Assume that H has a reproducing kernel $K(x, x')$. Let $g(x)$ be a degraded function of $f(x)$ which comes out from some measuring equipment. Let A_1 be the degradation operator which transforms $f(x)$ to $g(x)$. Assume that A1 is a linear bounded operator from H to H.

Let $\{x_m : 1 \leq m \leq M\} \subset D$ be a set of sample points which is not necessarily distributed uniformly in D. Let y be the M-dimensional vector consisting of sample values $\{g(x_m) : 1 \leq m \leq M\}$. Let A_2 be a sampling operator which transforms $g(x)$ to y. Let A be the observation operator defined by $A = A_2 A_1$ which transforms $f(x)$ to y.

The generalized sampling problem is to obtain the original function $f(x)$ or its best approximation from y. If $A_1 = I$ with I the identity operator on H, then it becomes the sampling theorem with ideal pulse. If $A_1 \neq I$, then it becomes the sampling theorem with real pulse. The following lemma is fundamental.

Lemma 2 *Let H_0 be a closed subspace of H and let P_0 be the orthogonal projection operator onto H_0. We can obtain the orthogonal projection of every $f \in H$ onto H_0 from $y = Af$ by using a linear operator X if and only if $H_0 \subset R(A^*)$.*

Proof. The operator equation $XA = P_0$ has a solution X if and only if $N(A) \subset N(P_0)$, which is equivalent to $H_0 \subset R(A^*)$. ■

Lemma 2 means that $R(A^*)$ is the largest subspace within which we can obtain the best approximation to the individual original $f \in H$ from y. We, therefore, concentrate on the maximal subspace $R(A^*)$ hereafter. Let P be the orthogonal projection operator onto $R(A^*)$ and let $f_0 = Pf$. The function f_0 is the best approximation to f in $R(A^*)$.

Theorem 14 *(Generalized sampling theorem) Let*

$$\phi_m^*(x) = K(x, x_m) \qquad : 1 \leq m \leq M, \tag{71}$$

$$\phi_m^*(x) = (A_1^*\phi_m^*)(x) \qquad : 1 \leq m \leq M. \tag{72}$$

Let $\{\phi_m : 1 \leq m \leq M\}$ be a counter-dual sequence to $\{\phi_m^* : 1 \leq m \leq M\}$ in $R(A^*)$. Then, we have

$$f_0(x) = \sum_{m=1}^{M} g(x_m)\phi_m(x). \tag{73}$$

Proof. First, we shall show that $\{\phi_m^* : 1 \leq m \leq M\}$ in eq.(72) spans $R(A^*)$. Since $\langle f, \phi_m^* \rangle = f(x_m)$ for f in H, we have

$$A_2 = \sum_{m=1}^{M} e_m \otimes \overline{\phi_m^*}. \tag{74}$$

Eqs.(74) yields

$$A = A_2A_1 = (\sum_{m=1}^{M} e_m \otimes \overline{\phi_m^*})A_1 = \sum_{m=1}^{M} e_m \otimes \overline{(A_1^*\phi_m^*)},$$

which implies

$$A = \sum_{m=1}^{M} e_m \otimes \overline{\phi_m^*}. \tag{75}$$

because of eq.(72). Eqs.(75) means that $\{\phi_m^* : 1 \leq m \leq M\}$ spans $R(A^*)$. Hence, we can construct $\{\phi_m : 1 \leq m \leq M\}$ from $\{\phi_m^* : 1 \leq m \leq M\}$. Eq.(72) yields

$$\langle f, \phi_m^* \rangle = \langle f, A_1^*\phi_m^* \rangle = \langle A_1 f, \phi_m^* \rangle = \langle g, \phi_m^* \rangle = g(x_m),$$

which implies eq.(73). ■

If $R(A^*) = H$, then $f_0 = f$ and eq.(73) restores the exact original function f. If $R(A^*) \subsetneqq H$, then eq.(73) provides the original f when $f \in R(A^*)$ and the best approximation to f when $f \notin R(A^*)$.

Let N be the dimension of $R(A^*)$. There exist an infinite number of counter-dual sequences if $M > N$. That means that there are an infinite number of restoration formulae (73) which provide the same function f_0. This is due to the linearly dependency of $\{\phi^s_m : 1 \leq m \leq M\}$. If we choose the sample points $\{x_m : 1 \leq m \leq M\}$ so that $\{\phi^*_m : 1 \leq m \leq M\}$ is linearly independent and if the degradation operator A_1 has a good property such as regurality, then $\{\phi^*_m : 1 \leq m \leq M\}$ is linearly independent and $\{\phi_m : 1 \leq m \leq M\}$ is uniquely determined. In that case, the PBOB $\{\phi_m, \phi^*_m : 1 \leq m \leq M\}$ becomes a BONB in $R(A^{]}*)$. Furthermore, if we choose the sample points $\{x_m : 1 \leq m \leq M\}$ so that $\{\phi^*_m : 1 \leq m \leq M\}$ is an orthogonal basis and if the degradation operator A_1 has a good property such as a unitary operator, then $\{\phi_m : 1 \leq m \leq M\}$ becomes an ONB in $R(A^*)$.

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